DISK TIDES AND ACCRETION RUNAWAY

William R. Ward[1]
Jet Propulsion Laboratory
California Institute of Technology
Pasadena, California 91109

and

Joseph M. Hahn
Department of Physics
University of Notre Dame
Notre Dame, Indiana 46556

Abstract: It is suggested that tidal interaction of an accreting planetary embryo with the gaseous pre-planetary disk may provide a mechanism to breach the so-called runway limit during the formation of the giant planet cores. The disk tidal torque converts a would-be shepherding object into a "predator", which can continue to cannibalize the planetesimal disk. This is more likely to occur in the giant planet region than the terrestrial zone, providing a natural cause for Jupiter to predate the inner planets and form within the $O(10^7)$ year lifetime of the nebula.

Subject headings: accretion - celestial mechanics - planets & satellites:general - solar system:formation

[1]Also at San Juan Capistrano Research Institute, San Juan Capistrano, CA

I. INTRODUCTION

One of the tightest constraints on models of solar system formation is the suspected 10^7 year lifetime of the solar nebula inferred from observations of T-tauri stars (Adams and Shu, 1986; Walter, 1986). The existence of gas giants like Jupiter and Saturn, establish that the planet building process for these objects was essentially completed before nebula dispersal. These planets are believed to have acquired their H/He component by gas accretion onto preexisting solid cores with estimated masses of 10-20 $M_\oplus$, where $M_\oplus$ - 6 X 10^{27} g denotes the mass of the earth (Mizuno, et al. , 1978: Bodenheimer and Pollack, 1986; Podolak, et al., 1993). If this model is correct, we must account for the accretion of $O(10^{29})$g cores within the lifetime of the gas disk.

II. RUNAWAY GROWTH

Current models of solid body accretion indicate that large embryos can form in a relatively short time scale due to the onset of accretion runaway (Greenberg et al., 1978; Wetherill and Stewart, 1986, 1993; Ida and Makino, 1993) . This runaway is due to a strong feed-back loop in the growth rate, M - $\sigma \Omega \pi R^2 F_g$, through the gravitational enhancement factor, F_g, where σ is the surface density of solid material in the disk, Ω is the embryo's mean motion, and M and R are its mass and radius, respectively (e.g., Greenzweig and Lissauer, 1990; Lissauer and Stewart, 1993). The enhancement factor is the ratio of the effective collision cross section to the geometrical cross section. If the relative velocities, v, are dominated by velocity dispersion instead of disk shear, the enhancement factor reads F_g = 1 + $(v_e/v)^2$, where $v_e = (2GM/R)^{1/2}$ is the embryo's escape velocity. Eventually, the embryo will grow large to stir the local planetesimal disk, (Lissauer,

1987).[2] This limits the enhancement factor to $F_g \sim (v_e/\xi L\Omega)^2 \approx 10^3 < ^{-2} (r/AU)$ and the characteristic growth time, $\tau \equiv R/\dot{R}$, becomes

$$(1) \qquad \tau \approx \frac{\rho_p R}{\sigma \Omega F_g} \approx 16\xi^2 (\frac{R}{km})(\frac{\rho_p}{\sigma})(\frac{r}{A.U.})^{1/2} \text{years}.$$

For values, $\sigma \sim 4$ g/cm^2, $\rho_p = 2$ g/cm^3, r ~ 5 A.U., $\xi \sim 4$, considered appropriate for the jovian zone, eqn (1) implies that a 15$M_\oplus$ giant planet core could accrete in $\lesssim 0(10^7)$ years.

However, there is an apparent obstacle to the formation of such full-sized planetary cores via runaway growth: that of local mass exhaustion. Dynamical friction tends to cause embryo orbits to become very circular during their growth (Greenberg, et al., 1978: Stewart and Wetherill, 1988; Wetherill and Stewart, 1989). There is critical value of the Jacobi constant below which a test particle cannot enter the Hill sphere of a object in a circular orbit (e.g., Hayashi et $al.$, 1977). This corresponds to a circular test body orbit with differential semi-major axes $\Delta a_c = 2\sqrt{3}L$, where $L \equiv r(M/3M_\odot)^{1/3}$ is the Hill sphere radius. This has often been interpreted as an "effective accretion range" for a growing embryo (Lissauer, 1987; Artymowicz, 1987; Wetherill and Stewart, 1989; Wetherill, 1990; Lissauer and Stewart, 1993). Although this range increases with the Hill radius, i.e., $\Delta a_c \propto M^{1/3}$; the width of the cannibalized zone should scale linearly with the mass. Thus , these widths become comparable when M = $2\pi\sigma r(2\Delta a_c)$, which implies a limiting runaway mass of

$$(2) \qquad M_R \sim \sqrt[4]{3}(8\pi\sigma r^2/M_\odot)^{3/2}M_\odot$$

At this point, the embryo is assumed to have largely isolated itself from the planetesimal disk so that runaway growth stalls. Further growth would have to rely on diffusion of planetesimals into the gap (Hayashi et al., 1977; Artymowicz, 1987) or on some other mechanism to supply new material such as gas drag (e.g. , Weidenschilling, 1977; Nakazawa and Nakagawa, 1982). The time scales for these processes are not given by eq (1) and can be much longer.

For the jovian zone, $M_R \approx 2M_\oplus$, which is about an order of magnitude too small for a giant planet core. It is the purpose of this letter to point out that this so-called runaway limit may be breached by strong disk-planet tidal interactions between the forming embryo and the remnant nebula that have not previously been included in these calculations.

III. TORQUE BALANCE

It is well known from planetary ring studies, that an object orbiting near a particle disk experiences a torque

$$(3) \qquad T = \pm C_d \mu^2 (\sigma r^2)(r\Omega)^2 \left(\frac{r}{w}\right)^3$$

that tends to repel the perturber from the disk, where w is the distance to the disk edge, r is the embryo's orbital radius, $\mu = M/M$. is its mass normalized to the primary, and C_d is a constant of order unity (e.g., Goldreich and Tremaine, 1979, 1980; Lin and Papaloizou, 1979).[3] If the embryo occupies a gap in a disk, it experiences a positive (negative) torque from the interior (exterior) portion of the disk. Acting alone,

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[3] A value of Cd $\approx$ 0.83 can be derived for a particle disk composed of nearly circular orbits, (e.g., Goldreich and Tremaine, 1982).

these torques keep the embryo roughly centered in the gap, i.e. , $T(w_i)$ + $T(w_o) = 0$, when $w_i \approx w_o$. However, in the presence of the nebula, the embryo experiences an additional tidal torque

(4) $$\Delta T_g = C_g \mu^2 (\sigma_g r^2) (r\Omega)^2 (\frac{r}{h})^2$$

due to its density wave interactions with the gaseous disk (e.g. , Goldreich and Tremaine, 1980 ; Ward, 1986, 1989 ; Korycansky and Pollack, 1993; Artymowicz, 1993). In eqn (4), σ_g represents the gas surface density, h - c/Ω is the scale height of the nebula, c is the gas sound speed, and C_g is a constant of order unity the exact value of which depends on the structural details of the nebula (see § V). As with ring torques, eqn (4) arises from the gravitational attraction of the embryo for spiral density waves that are launched at various Lindblad resonances between the perturber and gas disk (e.g., Goldreich and Tremaine, 1979a; Shu, 1984). Outer resonances exert negative torques on the perturber, inner resonances exert positive torques. The origin of net cumulative torque, ΔT_g , is a "mismatch" between the strengths of outer and inner Lindblad resonances due to global gradients of the disk (Goldreich and Tremaine, 1980; Ward, 1986). In addition, there is a torque contribution from coronation resonances that fall at the orbit of the perturber (e.g., Ward, 1993; Korycansky and Pollack, 1993). The sign of C_g is negative for most model calculations and we shall make that assumption here, although this is not crucial to our argument. This additional torque causes the embryo to be displaced inward from the center of the gap to occupy the torque balance position given by $T(w_i) + T(w_o) + \Delta T_g = 0$. This provides a constraint between the distance to the inner edge, W_i , and the gap's the "aspect" ratio, $w_i/w_o = \eta$,

(5) $\qquad c\left(\dfrac{w_i}{r}\right)^3 = \alpha\left(\dfrac{h}{r}\right)^2 (1 - \eta^3)$

where $\alpha = u/u_g$ is the solid/gas ratio and $C \equiv |C_g|/c_d$. In the limit $C_g \to 0$, $\eta \to 1$, $w_i = w_o$; but for $|C_g| > 0$, $\eta < 1$, $w_i < w_o$.

If most of the disk material within the boundaries of the gap is accreted by the embryo, its mass is $M \approx 2\pi\sigma r w_i (1+\eta^{-1})$. As before, the assumption is made that accretion runaway stalls when $W_i \geq A a_{c}$. Substituting Δa_c for W_i, yields

(6) $\qquad M = M_R \left(\dfrac{1 + \eta}{2\eta}\right)^{3/2}$

which can be combined with eq (5) to find

(7) $\qquad \left(1 - \eta^3\right)\left(\dfrac{2\eta}{1 + \eta}\right)^{3/2} = 8\sqrt{3}C\left(\dfrac{\mu_R}{\alpha}\right)\left(\dfrac{r}{h}\right)^2 \equiv K$

Fig. 1 shows the behavior of $\eta(K)$; the right-hand scale shows the corresponding value of $M(K)$ normalized to M_R. A maximum allowed K occurs at $dK/d\eta = 0$ for which $1-3\eta^3-2\eta^4 = 0$. This has the solution, $\eta_* = 0.618$ and a mass from eq (6) of $M_* = 1.498 M_R$. The value of K_* from eq (7) is $K(\eta_*) = 0.510$.

There are two branches to $\eta(K)$ for $K < K_*$; it is the upper branch that is relevant to the runaway process. The lower branch, at smaller η, corresponds to equilibrium states where the outer edge is increasingly remote and the torque balance is essentially between the nebula and inner disk tides only, For comparison, the curve, $K -- M_R/M = [2\eta/(1+\eta)]^{3/2}$, for a perturber orbiting just outside a disk, i.e. , $T(w_i = \Delta a_c) + A_g^T = 0$, is indicated by the dashed curve in Fig. 1.

IV. SHEPHERD OR PREDATOR

Torque balance with $W_i \geq \Delta a_c$ is not possible for $K > K_*$, which implies a critical value for the torque constant ratio

(8)
$$C_{crit} = \alpha K_* \frac{(h\ r)^2}{853 - \mu_R}$$

above which the embryo cannot isolate itself from the planetesimal disk. If the torque constant ratio, C, exceeds this threshold value, the nebula tidal torque will convert a would-be shepherding object into a "predator" that can continue to consume the planetesimal disk. Relaxing the constraint $W_i = \Delta a_c$, eqs (5) be used to find the "stand-off" distance, W_i , as a function of M/M_R for a given K,

(9)
$$M/M_R = \frac{1}{2}\left(\frac{W_i}{\Delta a_R}\right)\left\{1 + \left[1 - K\left(\frac{W_i}{\Delta a_R}\right)^3\right]^{-1/3}\right\}$$

where $\Delta a_R = \Delta a_c(M_R)$. Fig. 2 compares Δa_c with the stand-off distance given by eq (9) for several values of K. Note that as $M \gg M_R$, W_i approaches a constant value, $w_i = \Delta a_R/K^{1/3} = 4\sqrt[4]{3}(2\pi\sigma r^2/M_\odot)^{1/2} r/K^{1/3}$. This is because both T and ΔT_g are proportional to M^2 so that once the outer edge is remote, w_i must assume a constant value to balance the torques, independent of mass. It is clear that for $K > K_*$, w_i never exceeds Δa_c for any M so that isolation cannot occur. As the edge of planetesimal disk is stripped away, the nebula torque causes the embryo's orbit to decay, maintaining a small enough stand-off distance that the perturber can continue to accrete material.

So far, the effect of the reaction torque, $-T$, on the location of the disk particles has not been explicitly taken into account. For instance, it is well known that the shepherding action of a satellite on a planetary ring

can nudge material away from the satellite (Goldreich and Tremaine, 1979b) . However, a key difference between shepherding in a planetary ring vs a planetesimal disk is that in the latter, the optical depth, $\tau \sim \sigma/\rho_p R \approx 10^{-5} (R/km)^{-1}$, is low and collisions are much less frequent.' In this case, any epicyclic motion generated at an encounter does not damp out and particles repeatedly re-encounter the embryo with a non-zero eccentricity. In both ring and disk environments, particles suffer shepherding, i.e., a recoil of their semi-major axes away from the embryo. The crucial difference is that the Jacobi constant is generally decreased by particle collisions in a planetary ring, but is nearly conserved in tile planetesimal disk when such collisions are rare. If we interpret the half-width, w(J), appearing in eqn (3) as referring to the e = O (reference) orbit of those particles with the greatest Jacobi constant, then w(t) increases in a ring of high optical depth, but remains nearly constant in the planetesimal disk. Hence, the shepherding of semi-major axes in the planetesimal disk does not remove particles from the accretion zone. In this interpretation of equation (3), any effect of heating on the torque strength has been absorbed into the coefficient, $C_d(e,w)$, which is now to be considered a function of the particles' epicycle motions. Its behavior could be found by phase averaging numerical integrations of Hill's equations , We have not included such a procedure in this brief communication, but will report such results in a subsequent publication, Here we have been content to determine the critical torque coefficient in terms of the ratio $C \equiv |C_g| /C_d$.

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[4] Note that the collisional time scale, $\sim (\tau \Omega)^{-1}$, is longer than the synodic period at the Hill radius if $\tau << O(\mu^{1/3})$, which is easily satisfied by kilometer sized planetesimals in the vicinity of an earth mass embryo.

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V. TORQUE STRENGTH

In terms of the constants adopted for the jovian zone, $\alpha \equiv \sigma/\sigma_g - 10^{-2}$, h/r - .07, r - 5 AU, $\mu_R \sim 6.3 \times 10^{-6}$, the critical torque ratio is

(10)
$$C_{crit} - 0.29 \left(\frac{\alpha}{10^{-2}}\right) \left(\frac{T}{160°K}\right) \left(\frac{r}{5\ AU}\right)^{-2} \left(\frac{\sigma_d}{4\ g/cm^2}\right)^{-3/2}$$

By comparison, setting r = 1 A.U., $\sigma = 7\ g/cm^2$, T = $10^3\ °K$, α = .0036, as representative of the terrestrial region, eq (10) gives C_{crit} - 7.0. The increased threshold value, C_{crit}(1AU)/C_{crit}(5AU) - 24, in the terrestrial zone. implies that embryo isolation is much easier to achieve there.

A recent attempt t:o determine C_g in terms of the structural gradients of the nebula has been made by Korycansky and Pollack (1993) . Numerical integration of the 2D fluid equations for power law models of the form ρ_g a r-k, T a $r^{-\ell}$, yields, $C_g^L = -3.2(1+.28k+.81\ell)$ for Lindblad torques and $C_g^C = 2.0(1- .63k)$ for coronation torques, where only linear terms have been retained. The Lindblad value is consistent with an earlier semi-analytical estimate by Ward (1986). Both authors found that the keplerian rotation of the gas disk was a important source of differential torque. For disks with k, ℓ - $O(1)$, $|C_g|$ - $|C_g^L+C_g^C| \approx$ 6. Ward noted that these values, which pertain to a 2D disk, would likely be decreased by a finite thickness of the disk. This has been confirmed by Artymowicz (1993) who has used a vertically averaged potential together with an improved model for disk response at a Lindblad resonance to estimate the torque. Artymowicz's calculations indicate that vertical averaging causes nearly a 50% reduction in the differential torque.

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[5] Depending on phase, successive encounters can either increase or decrease e. From the Jacobi constant, changes in the semi-major axis obey $\delta a/a$ = - $\delta(e^2)\Omega/(\Omega-\Omega_s)$ and the angular momentum, H, lost by the particle per synodic period is $\Delta H/P_{syn} = \delta(e^2)(a\Omega)^2/4\pi = H$.

Interestingly, if $C_d \lesssim O(1)$, C could lie between the threshold values for the terrestrial and jovian regions for reasonable disk models. Although current calculations of the torques are not yet reliable enough to make such precise predictions, this problem makes it clear that further refinement of these calculations is a high priority issue.

VI. CONCLUSION

A failure of an embryo to achieve isolation may permit its continued growth toward the critical core size for gas accretion. The eventual onset of gas accretion will alter the local configuration of the gaseous nebula. Development of a low density zone in the gas disk surrounding the core should abort the nebula torque and stabilize the orbit. In addition, the protoplanet becomes a sink for any later clecaying embryos which attempt to follow the same evolutionary track - unless they are well enough spaced at the outset, e.g., a potential Saturnian core. A distinct advantage of this scheme is that it does not require a large excess of material to account for rapid core growth, most of which must later be removed from the system (Lissauer, 1987; Lissauer and Stewart, 1993). An intriguing additional possibility is that embryo isolation could still have occurred in the terrestrial zone, stalling the runaway process there. Growth to planetary size in the inner solar system could have proceeding along the slower, stochastic accretion route as described by Wetherill (1990). This implies that Jupiter may predate the terrestrial planets, a situation consistent with ideas concerning its role in the development of the asteroid belt (Wetherill, 1992). This also suggests that not all planetary systems will necessarily have giant planets. From eq (8), it is clear that small values of μ_R favor embryo isolation. Thus, small disks, (low mass and/or radius)

may not be conducive to rapid core formation and may tend to develop systems of smaller, more numerous planets,

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FIGURE CAPTIONS

Fig.1. Gap aspect ratio, $\eta \equiv w_i/l$, as a function of nebula torque strength, parametrized by $K = 8\,3C(\mu_R/\alpha)(r/h)^2$. Upper branch $(\eta > \eta_*)$ applies to runaway limit. Lower curve asymptotically approaches dashed curve for an object orbiting Aa outside a disk. Right-hand scale shows corresponding embryo mass normalized to value, M_R, when $K = 0$.

Fig.2. Distance from embryo to interior edge of gap, i.e., "stand-off distance", as a function of embryo mass for different values of K. Shown for comparison is accretion range $\Delta a \approx 2\sqrt{3}(M/3M_\odot)^{1/3}r$. Runaway growth stalls when $w_i \geq \Delta a_c$; this is possible 'only for $K \leq K_*$.
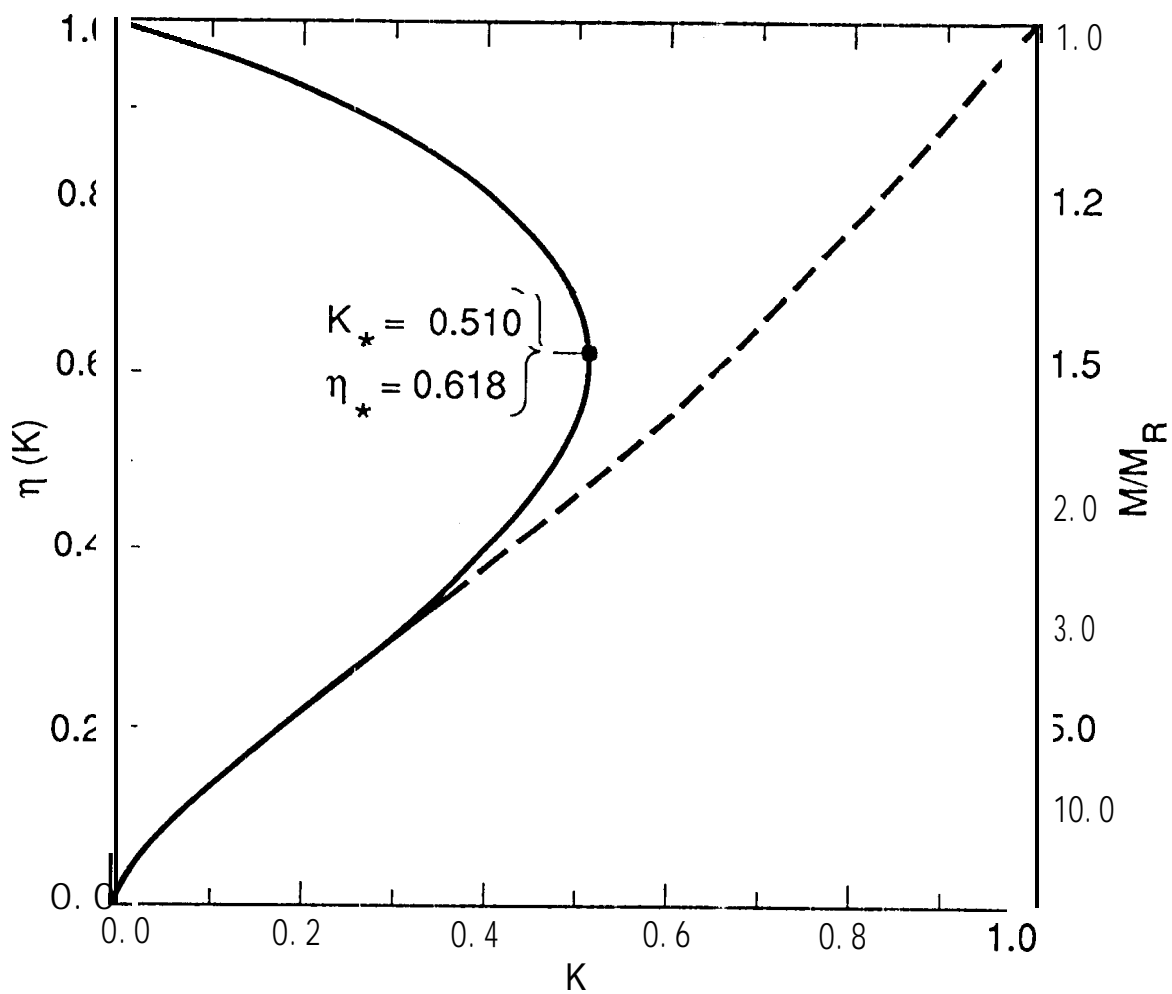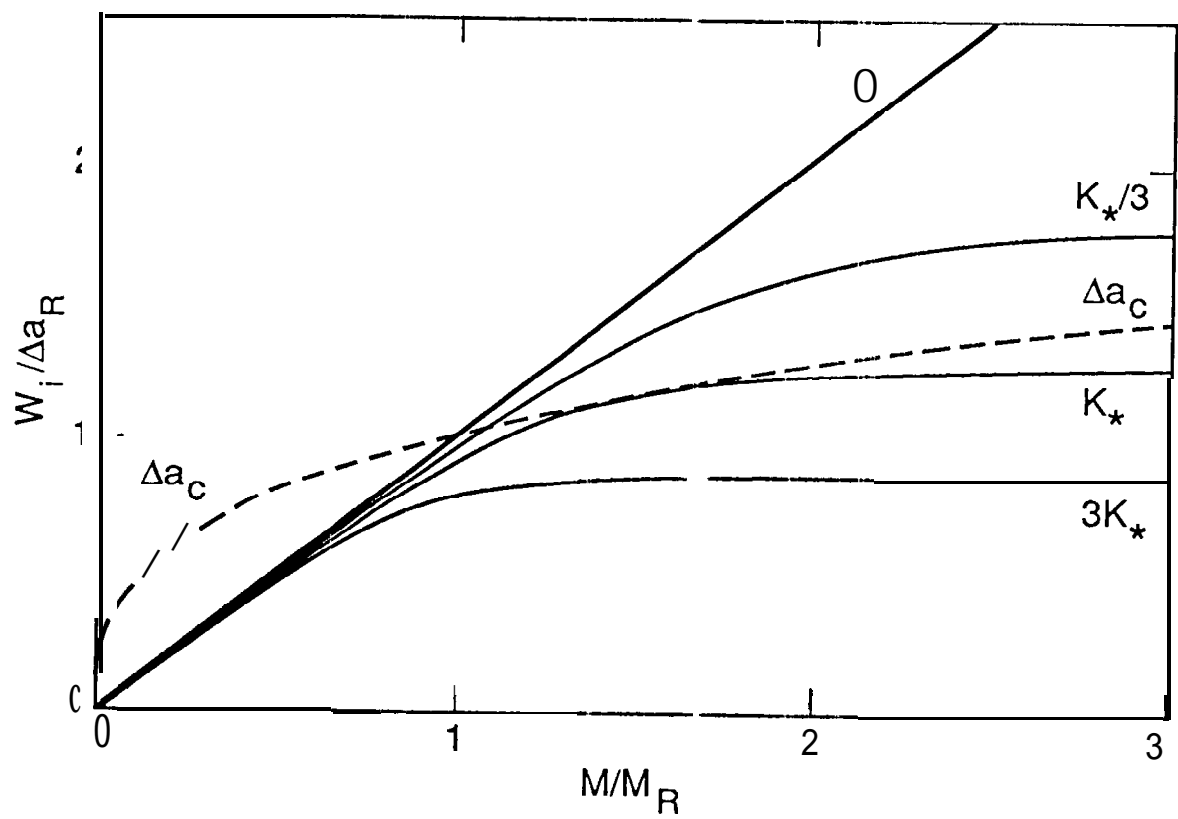
Figure 1

Figure 2